Solution by Roger A. Johnson, Western Reserve University.

Let $x = \angle EBG = \angle FBD$, $y = \angle GBF$, and $\alpha = \angle EBD$. Then $\angle BDF = 150° - \alpha/2$, and $\angle BGF = 90° - y/2$. Applying the law of sines,

$$\frac{\sin x}{\sin\left(150 - \frac{\alpha}{2}\right)} = \frac{FD}{BF}, \qquad \frac{\sin y}{\sin\left(90° - \frac{y}{2}\right)} = \frac{GF}{BF};$$

whence

$$\frac{\sin x}{\sin y} = \frac{\sin\left(150° - \frac{\alpha}{2}\right)}{\sin\left(90° - \frac{y}{2}\right)}. \tag{1}$$

Now in order that $x = y$, we must have either

$$150° - \frac{\alpha}{2} = 90° - \frac{y}{2}, \tag{a}$$

or

$$150° - \frac{\alpha}{2} = 180° - \left(90° - \frac{y}{2}\right). \tag{b}$$

Letting $y = \alpha/3$, we obtain (*a*) $\alpha = 180°$, (*b*) $\alpha = 90°$, as the only values of α for which this construction effects the trisection. Conversely, it is very easily seen directly that either a right angle or a straight angle is actually trisected.

To determine roughly the magnitude of the error for angles in general, we may safely replace $\sin(90° - y/2)$ by $\sin(90° - \alpha/6)$; we have then

$$\frac{\sin x}{\sin y} = \frac{\sin\left(30° + \frac{\alpha}{2}\right)}{\sin\left(90° - \frac{\alpha}{6}\right)}. \tag{2}$$

For small angles, the construction is far from accurate; in fact,

$$\lim_{\alpha=0} \frac{\sin x}{\sin y} = \frac{\sin 30°}{\sin 90°} = \frac{1}{2},$$

so that if α is small, it is divided approximately in *fourths*. We find roughly, when $\alpha = 30°$, $60°$, $90°$, $120°$, $150°$, and $180°$, $(\sin x)/(\sin y) = .710, .879, 1.000, 1.064, 1.066$, and 1.000, respectively. Again, by setting $x = \alpha/2 - y/2$ in (1) and solving for $y/2$, we find

$$\cot\frac{y}{2} = 2\cot\frac{\alpha}{2} + \sqrt{3};$$

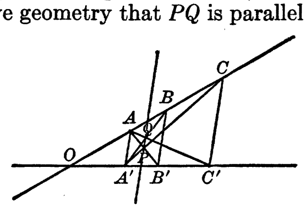
whence we may compute the value of y corresponding to any value of α. For example, if $\alpha = 10°$, $y = 4°\,40'$ and $x = 2°\,40'$; if $\alpha = 60°$, $y = 21°\,46'$ and $x = 19°\,7'$; if $\alpha = 135°$, $y = 42°\,41'$ and $x = 46°\,10'$.

Also solved by Horace Olson.

500. Proposed by R. T. McGregor, Bangor, California.

OABC, *OA'B'C'* are two straight lines such that *AA'*, *BB'*, *CC'* are parallel. *AB'*, *A'C* meet in *P*; *A'B* and *AC'* meet in *Q*. Show by synthetic projective geometry that *PQ* is parallel to *AA'*. Milne's *Projective Geometry*, Chap. I, Ex. 20.

Solution by Claribel Kendall, University of Colorado.



Consider A, B, C as the first, third and fifth sides, respectively of a hexagon; and A', B', C' as the fourth, second, and sixth sides, respectively, of the same hexagon. (12), (45); (34), (61) meet in P and Q, respectively; hence, (23), (56) must meet in a point collinear with P and Q.

This follows as a special case of Pascal's Theorem for a hexagon inscribed in a conic.* The conic is in this case degenerate, two straight lines. But (23), (56) are parallel and hence must have an infinitely distant point in common with PQ. Therefore PQ is parallel to AA'.

Also solved by MARJORIE L. BROWN, O. S. ADAMS (two methods), F. E. WOOD, NATHAN ALTSHILLER, HANNAH SUFFIN, and H. H. CONWELL.

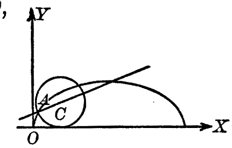
CALCULUS.

416. Proposed by CHARLES N. SCHMALL, New York City.

If A be a point on a cycloid and C the corresponding position of the center of the generating circle, show that AC envelops another cycloid half the size of the first.

SOLUTION BY A. M. HARDING, University of Arkansas.

The coördinates of any point on the cycloid are $x = a\theta - a\sin\theta$, $y = a - a\cos\theta$, and the coördinates of the center of the generating circle are given by $x = a\theta$, $y = a$. The equation of AC is



$$\frac{x - a\theta}{\sin\theta} = \frac{y - a}{\cos\theta},$$

or

$$y - a = \cot\theta(x - a\theta).$$

The equation of a cycloid half the size of the given cycloid and having a cusp at O is

$$x = \frac{a\varphi}{2} - \frac{a}{2}\sin\varphi, \qquad y = \frac{a}{2} - \frac{a}{2}\cos\varphi.$$

We propose to show that AC is always tangent to this cycloid.

The equation of any tangent to this cycloid is

$$y - \frac{a}{2}(1 - \cos\varphi) = \cot\frac{\varphi}{2}\left[x - \frac{a}{2}(\varphi - \sin\varphi)\right].$$

Let $\varphi = 2\theta$. This equation then becomes

$$y - a\sin^2\theta = \cot\theta[x - a\theta + a\sin\theta\cos\theta]$$

or

$$y - a = \cot\theta(x - a\theta),$$

which is the same as the equation of AC.

Also solved by C. N. SCHMALL, ELIJAH SWIFT, G. W. HARTWELL, HORACE OLSON, O. S. ADAMS, M. R. GAFFET, R. H. HOWARD, J. B. REYNOLDS, and SHIMPEI NISHIMURA.

417. Proposed by H. S. UHLER, Yale University.

To the degree of approximation indicated show that $(\sqrt{-1})^{\sqrt{-1}} = 0.207879576351$.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

It is not difficult to show, as required in Todhunter's *Plane Trigonometry*, Ed. 1913, pp. 320–21, Examples 266, 275, "that $(a + bi)^{\alpha+\beta i}$ will be wholly real or imaginary if

$$(\beta/2)\log(a^2 + b^2) + \alpha\tan^{-1}(b/a)$$

is (I) zero, or an even multiple of $\pi/2$; or (II) an odd multiple $\pi/2$". In the problem, $a = \alpha = 0$, and $b = \beta = 1$, the conditions corresponding to (I), 0 being called an even number. $(\sqrt{-1})^{\sqrt{-1}} = e^{-\frac{1}{2}\pi}$, and the numerical value can be tested by using enough decimals in the values of e and π.

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* Cremona, *Elements of Projective Geometry*, translated by Leudensdorf, Articles 88 and 153, 3d edition.